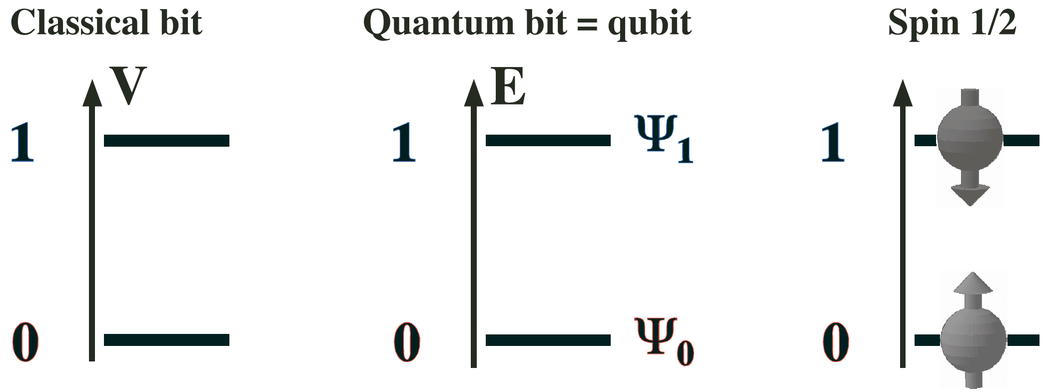
INFORMATION TRANSPORT IN SPIN CHAINS

Chains of coupled spins 1/2 (or qubits) are potentially useful for the short-range transport of information in a quantum computing device. Of special importance is the transport of entanglement, i.e. the undistorted transfer of quantum states involving more that one qubit. The earliest proposals for information transport involved standard spin waves which, however, inevitably show dispersion effects and consequently distort any initial wave packet. Later it was discovered that XX chains with site-dependent couplings or magnetic fields allow for perfect transfer of quantum states. Those systems were found by mapping a chain of N spins 1/2 to a single spin N/2 in a transverse magnetic field. We showed how to generalize the original proposal which involved strongly varying values of the couplings. In the generalized scheme perfect transfer of states is achieved even in systems with almost homogeneous couplings. I will also discuss alternative concepts and more recent developments.

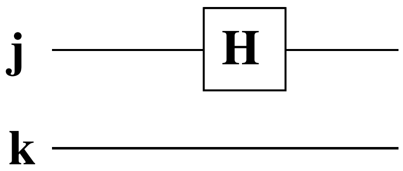
Quantum hardware

Quantum bits store information. Superpositions $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \rightarrow$ quantum parallelism.



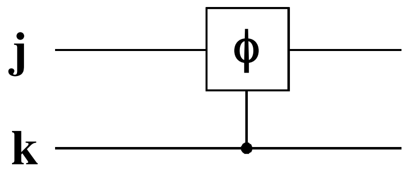
Quantum gates manipulate information

Single-qubit gate



$$U_j = e^{i\frac{\pi}{2\sqrt{2}}(X_j+Z_j)}$$

Two-qubit gate

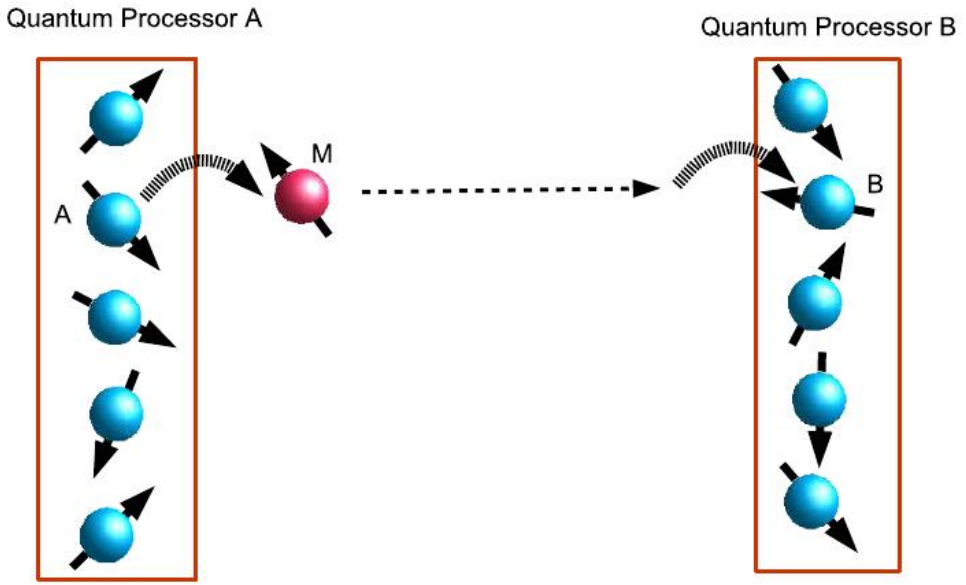


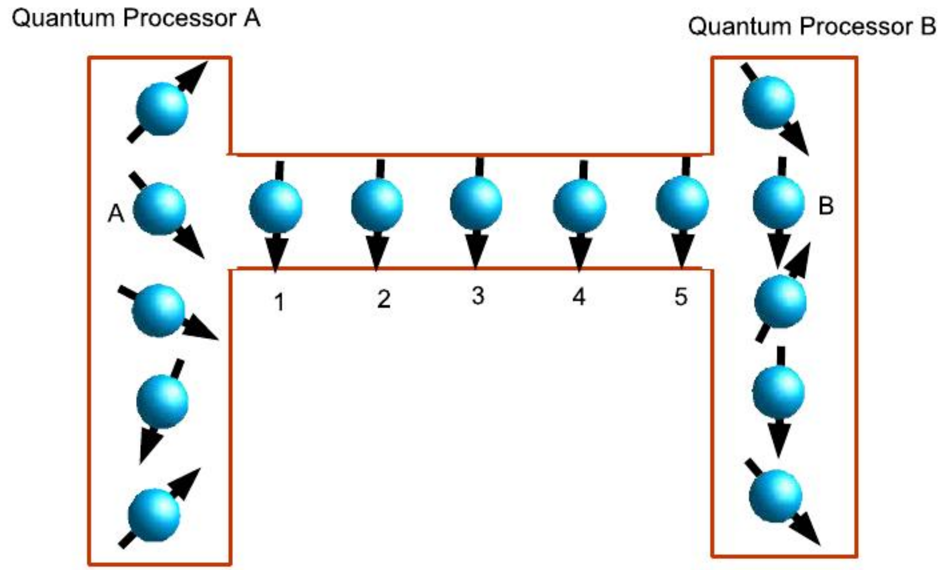
$$U_{jk} = e^{i\phi(Z_j+Z_k-Z_jZ_k)}$$

What about quantum lines to transmit information?

Short-range transfer of multi-qubit states?

Single photons carry no entanglement, but quantum algorithms must handle entangled states.





Within a quantum processor these states must be transferred between quantum registers or from/to quantum memory, over short distances. Spin chains may be useful for that purpose.

Pictures and quotation from S. Bose: *Quantum Communication Through Spin Chain Dynamics: An Introductory Overview*, Contemp. Phys. **48**, 13-30 (2007); arXiv:0802.1224.

This review article will be based on one such alternative, where the quantum state transfer is accomplished purely through the natural dynamical evolution of a permanently coupled chain of quantum systems, which has recently drawn considerable attention [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63].

The spin-$\frac{1}{2}$ Heisenberg-XXZ chain

Heisenberg exchange interaction between two $s = \frac{1}{2}$ spins

$$H_{\text{Heisenberg}} = -J\vec{S}_1 \cdot \vec{S}_2$$

chain of N spins with nearest-neighbor interactions, anisotropic in spin space:

$$\begin{aligned} H_{\text{XXZ}} &= -J\sum_{i=1}^{N-1}\left[(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \Delta S_i^z S_{i+1}^z\right] - h\sum_{i=1}^{N} S_i^z \\ &= -J\sum_{i=1}^{N-1}\left[\frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + \Delta S_i^z S_{i+1}^z\right] - h\sum_{i=1}^{N} S_i^z \end{aligned}$$

Jordan-Wigner mapping $\triangleright$

Spins	$\longleftrightarrow$	Fermions
S^+, S^-	$\longleftrightarrow$	$\pm a^\dagger, \pm a$
S^z	$\longleftrightarrow$	$a^\dagger a - 1/2$
$J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)$	$\longleftrightarrow$	$t\ (a_i^\dagger a_{i+1} + \text{h.c.})$ hopping
$\Delta J\ S_i^z S_{i+1}^z$	$\longleftrightarrow$	$V\ n_i n_{i+1}$ *interaction*
$h\ S_i^z$	$\longleftrightarrow$	$\mu\ n_i$ chemical potential

The spin-$\frac{1}{2}$ Heisenberg-XXZ chain: Eigenstates

The ferromagnetic ground state: $|\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow\rangle = |000 \ldots 000\rangle$.

A single spin-flip state $S_2^- |\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow\rangle = |\uparrow\downarrow\uparrow \ldots \uparrow\uparrow\uparrow\rangle = |010 \ldots 000\rangle$
is not an eigenstate of H_{XXZ}: $(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+)$ *moves* the inverted spin left or right.

How about coherent transport ?

A single spin-wave state

$$|q\rangle = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} e^{iqr} S_r^- |\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow\rangle =: S^-(q) |\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow\rangle$$

is an eigenstate of H_{XXZ} with energy $\hbar\omega(q) = -J \cos q$. In the Jordan-Wigner picture this corresponds to a single fermion in a Bloch state in a tight-binding chain model.

However, a two spin-wave state

$$S(q_1)^- S(q_2)^- |\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow\rangle$$

is not an eigenstate of H_{XXZ}: the Jordan-Wigner fermions interact due to the $S_i^z S_{i+1}^z$ term.

Undistorted transfer of states with two or more flipped spins is probably difficult due to interaction. Unfortunately single spin-flip states have problems of their own...

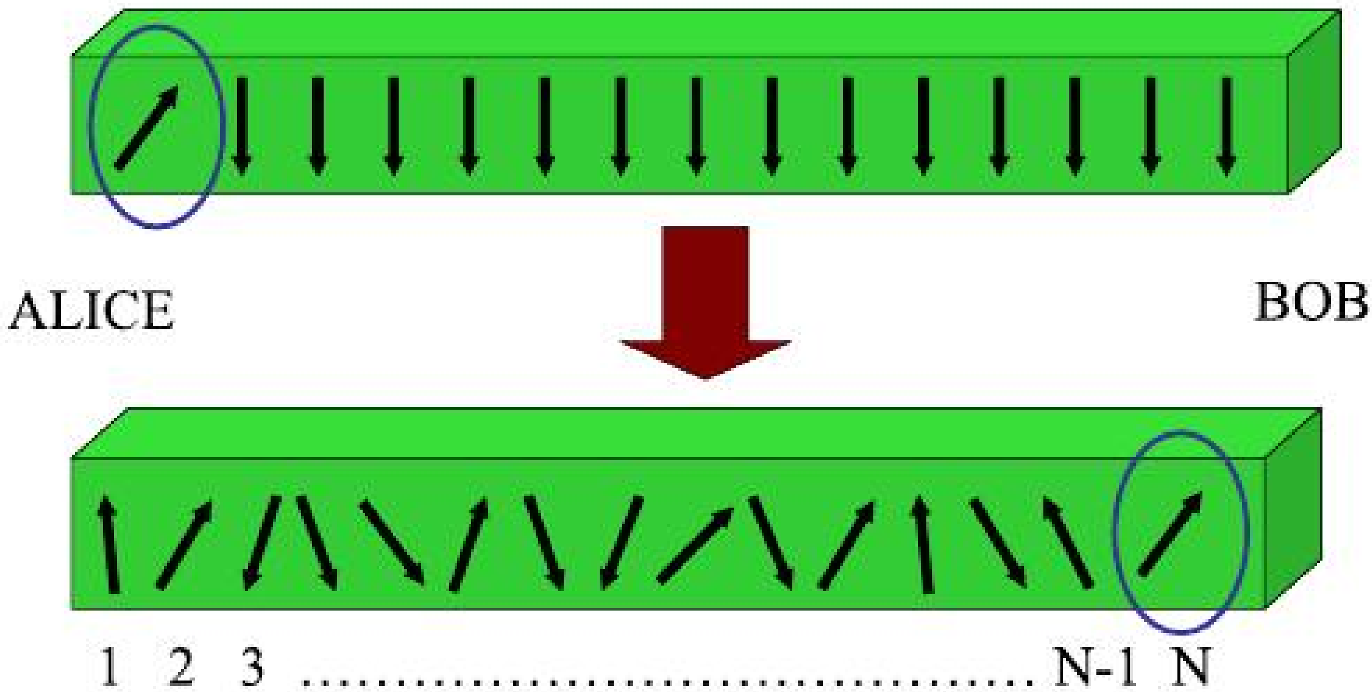
Spin wave packets

S. Bose: *Quantum communication through an unmodulated spin chain* PRL **91**, 207901 (2003)

Prepare the first spin of a Heisenberg chain in a superposition state $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$:

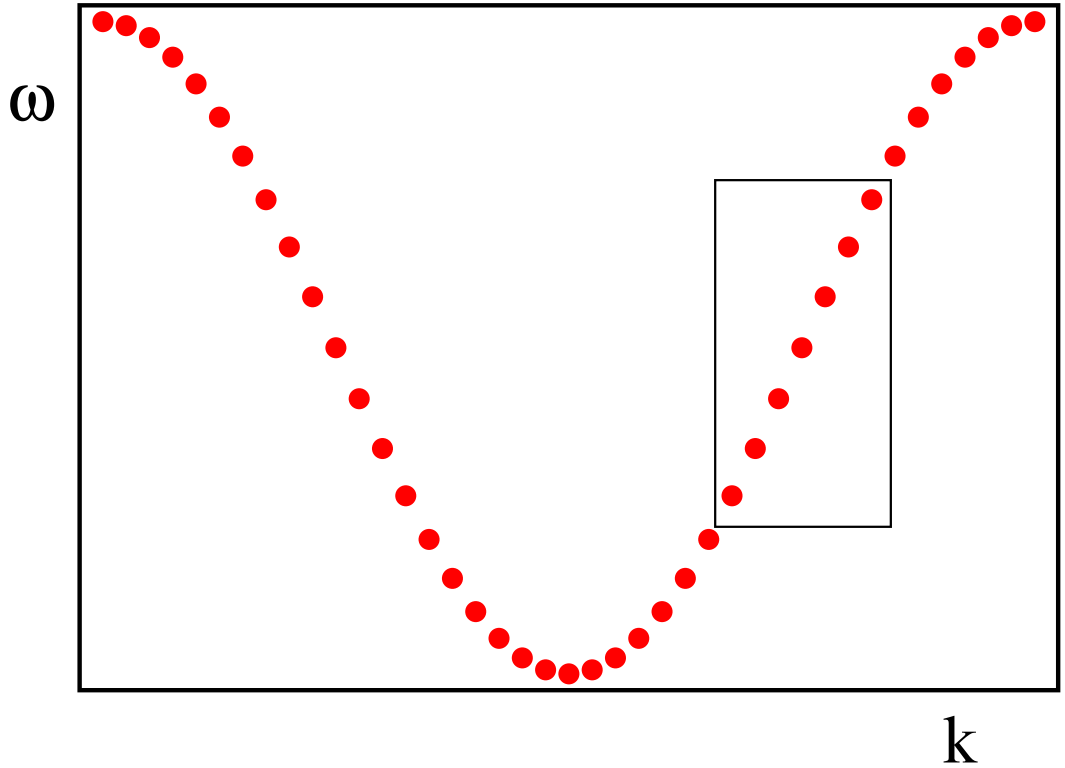
$$(\alpha|\uparrow\rangle + \beta|\downarrow\rangle) \otimes |\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow\rangle$$

is a superposition of the ground state and of single spin-wave states: a spin wave packet which may be received with reasonable fidelity at the other end of the chain after a certain time.



T.J. Osborne and N. Linden: *Propagation of quantum information through a spin system* PRA **69**, 052315 (2004)

Instead of states localized at a single site, transfer Gaussian spin wave packets which occupy *only the least dispersive part of the dispersion relation*, and which are narrow in wavevector space rather than in real space.



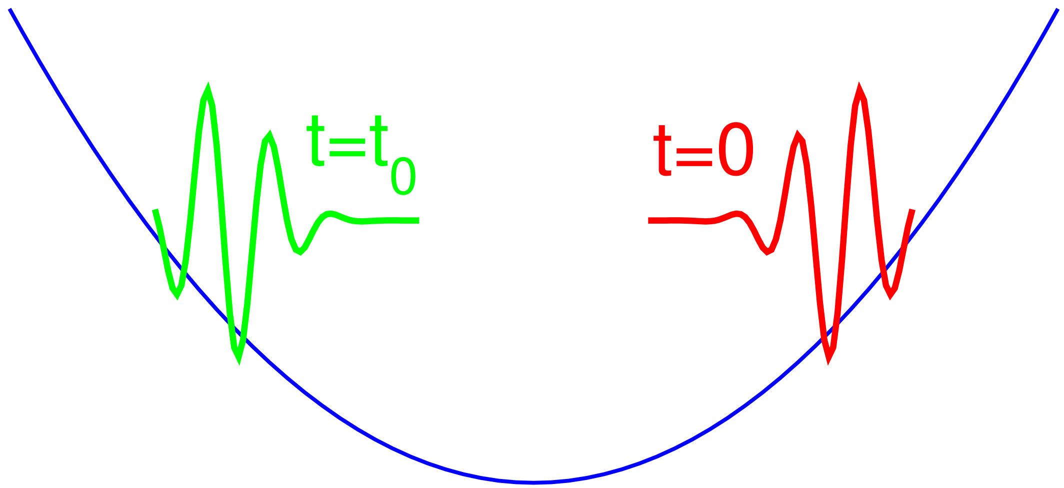
Note:
Least dispersive ≈ linear $\omega(k)$
≈ equidistant energy values

→ fairly good transfer of wave packets.

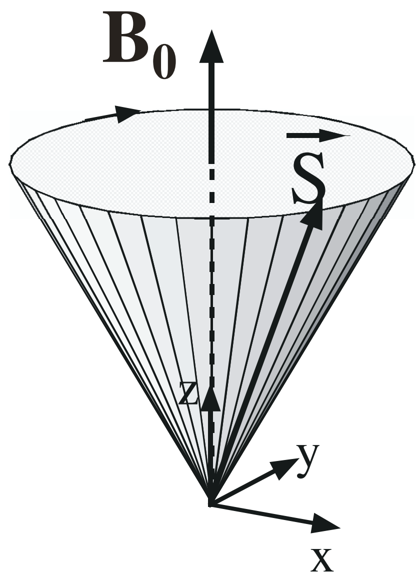
Fairly good is fine, but...

How about perfect transfer ?

Harmonic oscillator: Any wavepacket initially localized on the right develops into its perfect mirror image localized on the left.
Equidistant spectrum, but continuous degrees of freedom. Difficult to define qubits.
(Possible, though: continuous-variable quantum computing)



Angular momentum J: $|J_z = +J\rangle$ can develop into $|J_z = -J\rangle$ by rotation in a transverse field.
Equidistant spectrum, but zero-dimensional.
"Transport" in Hilbert space,
no transport in real space.



M. Christandl et al.: *Perfect State Transfer in Quantum Spin Networks* PRL **92**, 187902 (2004)
C. Albanese et al.: *Mirror Inversion of Quantum States in Linear Registers* PRL **93**, 230502 (2004).

Single particle on a $(2J+1)$-site chain $\Longleftrightarrow$ Angular momentum J

State $|n\rangle$ localized at lattice site $n = 1, ..., 2J+1$ $\Longleftrightarrow$ J_z eigenstate $|m\rangle$

Transition amplitude (hopping matrix element) between n and $n \pm 1$ $\Longleftrightarrow$ (J_x or J_y) matrix element between $|m\rangle$ and $|m \pm 1\rangle$.

$$2J_x|m\rangle = (J_+ + J_-)|m\rangle = \sqrt{(J+m+1)(J-m)}|m+1\rangle + \sqrt{(J+m)(J-m+1)}|m-1\rangle$$

Find a lattice Hamiltonian H such that

$$H|n\rangle = \sqrt{n(N-n)}|n+1\rangle + \sqrt{(n-1)(N-n+1)}|n-1\rangle.$$

Solution

$$\begin{aligned} H &= \sum_{n=1}^{N-1} \sqrt{n(N-n)} \left[\frac{1}{2}\left(a^\dagger_{n+1}a_n + hc\right) + \Delta\left(a^\dagger_n a_n - \frac{1}{2}\right)\left(a^\dagger_{n+1}a_{n+1} - \frac{1}{2}\right)\right] \\ &\overset{JW}{=} \sum_{n=1}^{N-1} \sqrt{n(N-n)} \left[(S^x_n S^x_{n+1} + S^y_n S^y_{n+1}) + \Delta S^z_n S^z_{n+1}\right] \end{aligned}$$

$$H = \sum_{n=1}^{N-1} \sqrt{n(N-n)}\,\left[(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) + \Delta S_n^z S_{n+1}^z\right]$$

Inhomogeneous XXZ chain; however, Δ causes problems even if only one particle is present.
(M. Wieśniak, *Heisenberg chains cannot mirror a state* Phys. Rev. A **78** 052334 (2008))

Consider $\Delta = 0$ from now on $\rightarrow$ inhomogeneous XX chain.

A state $|x\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ of spin 1 is transferred to spin N by H after a time τ:

$$|x\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle \quad \longrightarrow \quad |\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow x\rangle$$

This is still just single-qubit transport; however, after the same time τ

$$|\uparrow x\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle \quad \longrightarrow \quad |\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow x\uparrow\rangle$$

and also

$$|\uparrow\uparrow x\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle \quad \longrightarrow \quad |\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow x\uparrow\uparrow\rangle$$

.... and so on.

The Hamiltonian mirrors states

Every state $|xyz\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle$ of the spin chain is mapped to its mirror image $|\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow zyx\rangle$ after τ, but only for $\Delta = 0$, so that "particles" (reversed spins) do not interact with each other (and do not feel an inhomogeneous magnetic field),
$\longrightarrow$ inhomogeneous XX chain.

The mirror property of this spin-$\frac{1}{2}$ chain is due to
- the equidistant energy spectrum
- symmetry properties of the corresponding eigenvectors.

There is another spin-$\frac{1}{2}$ chain which acts as a perfect mirror for states (inhomogeneous XX with additional field in z direction).

Some simple questions

- Is that all or are there more mirror chains?
- If there are, can we engineer chains with perfect transfer/mirroring properties, plus other desirable features?
- How about mixed $(T > 0)$ states?
- What is really needed to achieve perfect transfer ?

Back to the model

General inhomogeneous open-ended $(N+1)$-site $S = \frac{1}{2}$ XX chain:

$$H = 2\sum_{i=1}^{N} J_i(S_i^x S_{i-1}^x + S_i^y S_{i-1}^y) + \sum_{i=0}^{N} h_i \left(S_i^z + \frac{1}{2}\right).$$

Equivalent Hamiltonian of noninteracting "spinless" lattice fermions:

$$H = \sum_{i=1}^{N} J_i(c_{i-1}^\dagger c_i + c_i^\dagger c_{i-1}) + \sum_{i=0}^{N} h_i c_i^\dagger c_i$$

can be diagonalized,

$$H = \sum_{\nu=0}^{N} \varepsilon_\nu c_\nu^\dagger c_\nu.$$

$c_\nu^\dagger$ creates a fermion in a single-particle eigenstate $|\nu\rangle$ of energy ε_ν;
$c_i^\dagger$ creates a fermion at lattice site i.

The ε_ν and $|\nu\rangle$ determine the dynamics completely: every eigenstate of H is uniquely characterized by the fermion occupation numbers $n_\nu = c_\nu^\dagger c_\nu$.

Single-particle properties of a mirror Hamiltonian

ε_ν $(\nu = 0, ..., N)$ and $|\nu\rangle$ are eigenvalues and eigenvectors of the one-particle Hamiltonian matrix H_1.

Mirror symmetry: $h_i = h_{N-i}$ and $J_i = J_{N+1-i}$

$\Rightarrow$ the eigenvectors of H_1, have definite parity: either $\langle i|\nu\rangle = + \langle N-i|\nu\rangle$ or $\langle i|\nu\rangle = -\langle N-i|\nu\rangle$.

Parity alternates as ε_ν grows.

$$H_1 = \begin{pmatrix} h_0 & J_1 & & & & \\ J_1 & h_1 & J_2 & & & \\ & J_2 & h_2 & J_3 & & \\ & & J_3 & \ddots & & \\ & & & & \ddots & J_N \\ & & & & J_N & h_N \end{pmatrix}$$

(Discrete version of the "Knotensatz": *For a real symmetric tridiagonal matrix with only positive subdiagonal elements (i) all eigenvalues are real and nondegenerate, and (ii) the sequence of the components of the jth eigenvector ,in ascending order of the eigenvalues, $j = 0, 1, \ldots$ shows exactly j sign changes.*)

The eigenvectors of H_1, i.e. the single-particle eigenstates of H, are alternately even and odd.

Wanted: Operation M which maps an arbitrary many-particle state to its spatial mirror image.

Sufficient: M maps every single-particle state $|\nu\rangle$ to its mirror image:

$$M = \Pi(-1)^\nu \qquad (\Pi\text{: parity, i.e. } k \to N-k).$$

Implement the extra sign for the odd states as a dynamical phase factor $\exp[i\pi(2n+1)]$ by designing the ε_ν appropriately.

Designing the spectrum

Evolution of the single-particle state $|i\rangle$ localized at site i:

$$e^{-iHt}|i\rangle = \sum_{\nu=0}^{N} e^{-i\varepsilon_\nu t}|\nu\rangle\langle\nu|i\rangle.$$ This is what we have.

Alternating parity $\Rightarrow$ $\langle N-i|\nu\rangle = (-1)^\nu\langle i|\nu\rangle$ $\Rightarrow$

$$|N-i\rangle = \sum_\nu |\nu\rangle\langle\nu|N-i\rangle = \sum_\nu (-1)^\nu|\nu\rangle\langle\nu|i\rangle.$$ This is what we want.

$\Rightarrow$ This is what we need:

$$e^{-i\varepsilon_\nu\tau} = (-1)^\nu e^{i\phi_0} = e^{-i(\pi\nu-\phi_0)}$$

or equivalently

$$\varepsilon_\nu\tau = (2n(\nu)+\nu)\pi - \phi_0,$$

where $n(\nu)$ is an *arbitrary* integer function.

Every system with such single-particle energies generates perfect mirror images of arbitrary input states!

Designing the Hamiltonian ?

The function $n(\nu)$ in $\varepsilon_\nu \tau = (2n(\nu) + \nu)\pi - \phi_0$ is completely arbitrary $\Rightarrow$ infinitely many single-particle spectra suitable for quantum state mirroring.
$n(\nu) \equiv 0$ and $n(\nu) = q\frac{\nu(\nu+1)}{2} + p\nu \quad (p, q$ integer) are the systems of Albanese et al.

Which Hamiltonian (if any) yields a given / desired spectrum ε_ν ?

Hald 1976: For a given nondegenerate (single-particle) spectrum there exists a unique symmetric tridiagonal Hamiltonian matrix with nonnegative subdiagonal elements and with the additional spatial symmetry properties discussed above.

How to find that matrix ?

- Direct method; algorithms by Hochstadt (1974), Sussman-Fort (1982), Wang et al. (2011), Bruderer et al. (2012).

- Simulated annealing: optimizing the set of eigenvalues.

What do we have ?

Many proposals for quantum information transfer in spin chains are restricted: a single spin state is transported through the completely polarized (ground) state.

Here, states involving arbitrarily many sites are perfectly mirrored across the system.
No restriction to the ground state nor even to the set of pure states.
(All single-fermion eigenstates of the Hamiltonian and thus arbitrary many-fermion density operators are mirrored perfectly at the same instant of time τ.)

Mirroring twice reproduces the initial state.

$\Rightarrow$ Time evolution of the system is periodic with period 2τ.

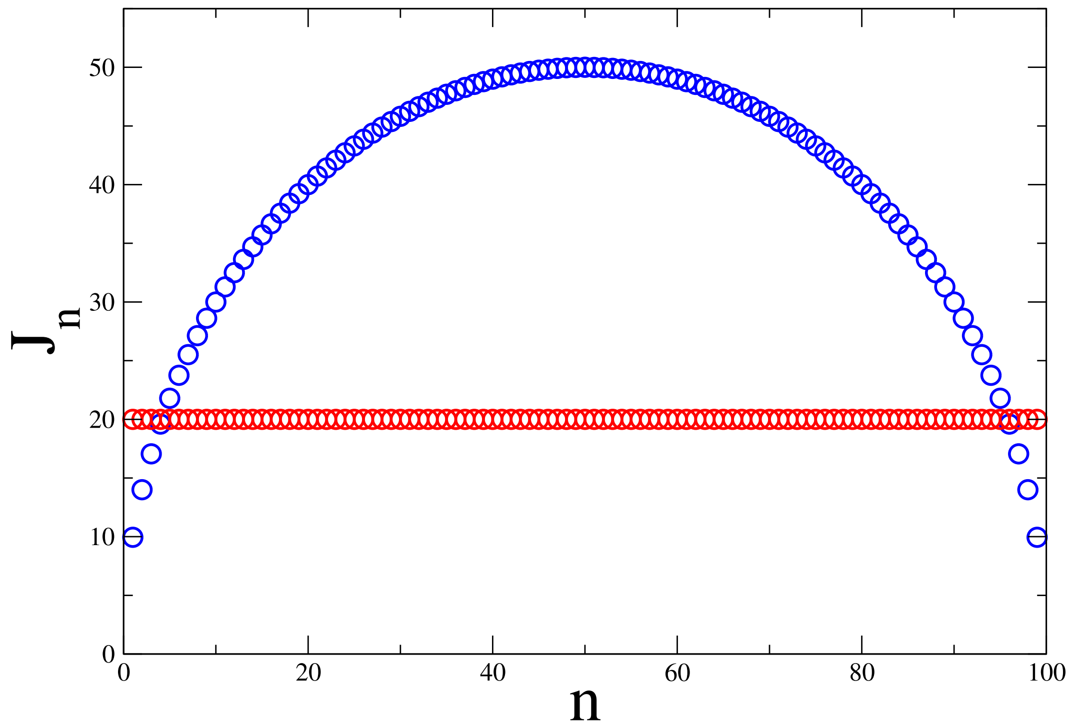
Proof: Time autocorrelation function of an arbitrary observable $A = A^\dagger$:

$$\langle A(t)A \rangle = Z^{-1} \sum_n \langle n | e^{-\beta H} e^{iHt} A e^{-iHt} A | n \rangle = Z^{-1} \sum_{n,m} e^{-\beta E_n} e^{i(E_n - E_m)t} |\langle n | A | m \rangle|^2$$

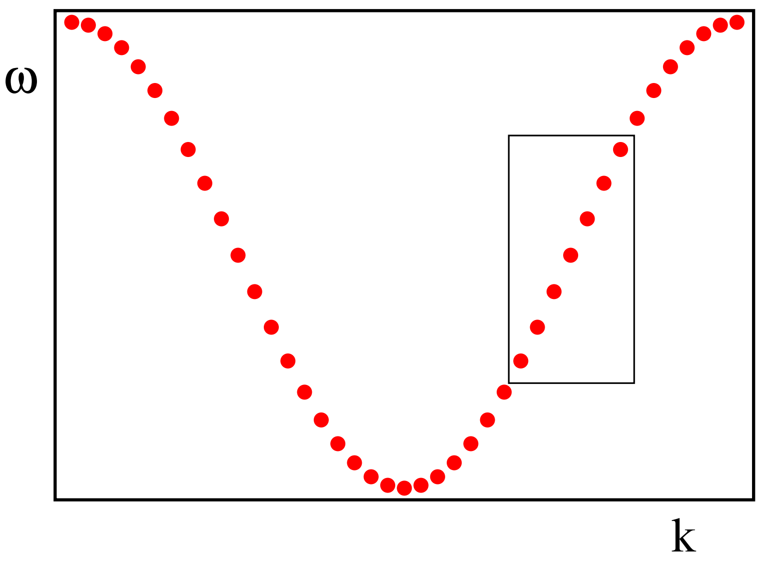
$(Z = \sum_n e^{-\beta E_n} \quad ; \quad \beta = (k_B T)^{-1} \quad ; \quad H|n\rangle = E_n |n\rangle)$
$(E_n - E_m)$ are all multiples of some energy, $\Rightarrow \langle A(t)A \rangle$ is a periodic function of t.

Quantum spin chain engineering

Homogeneous XX chain:
simple, but no perfect transport (dispersion).
Inhomogeneous chain:
Perfect transport, but awkward couplings.



Compromise ?

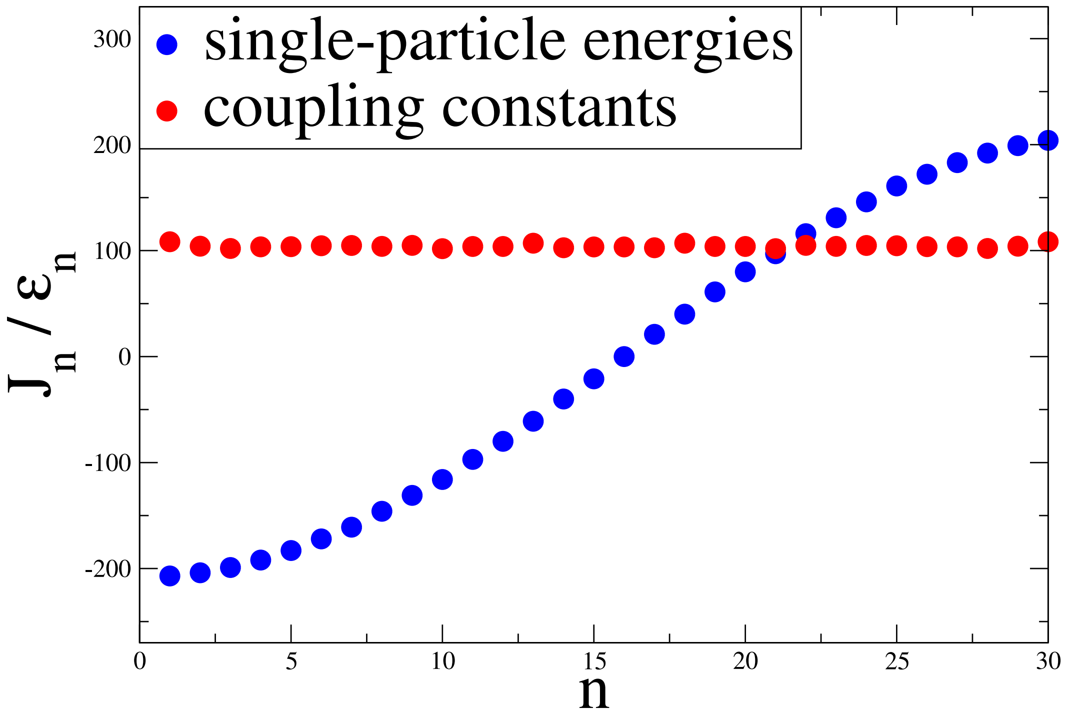


Idea:
Bring the old spin-wave dispersion relation into the right shape (all energy differences are suitable multiples of something) by a little *tweaking*.

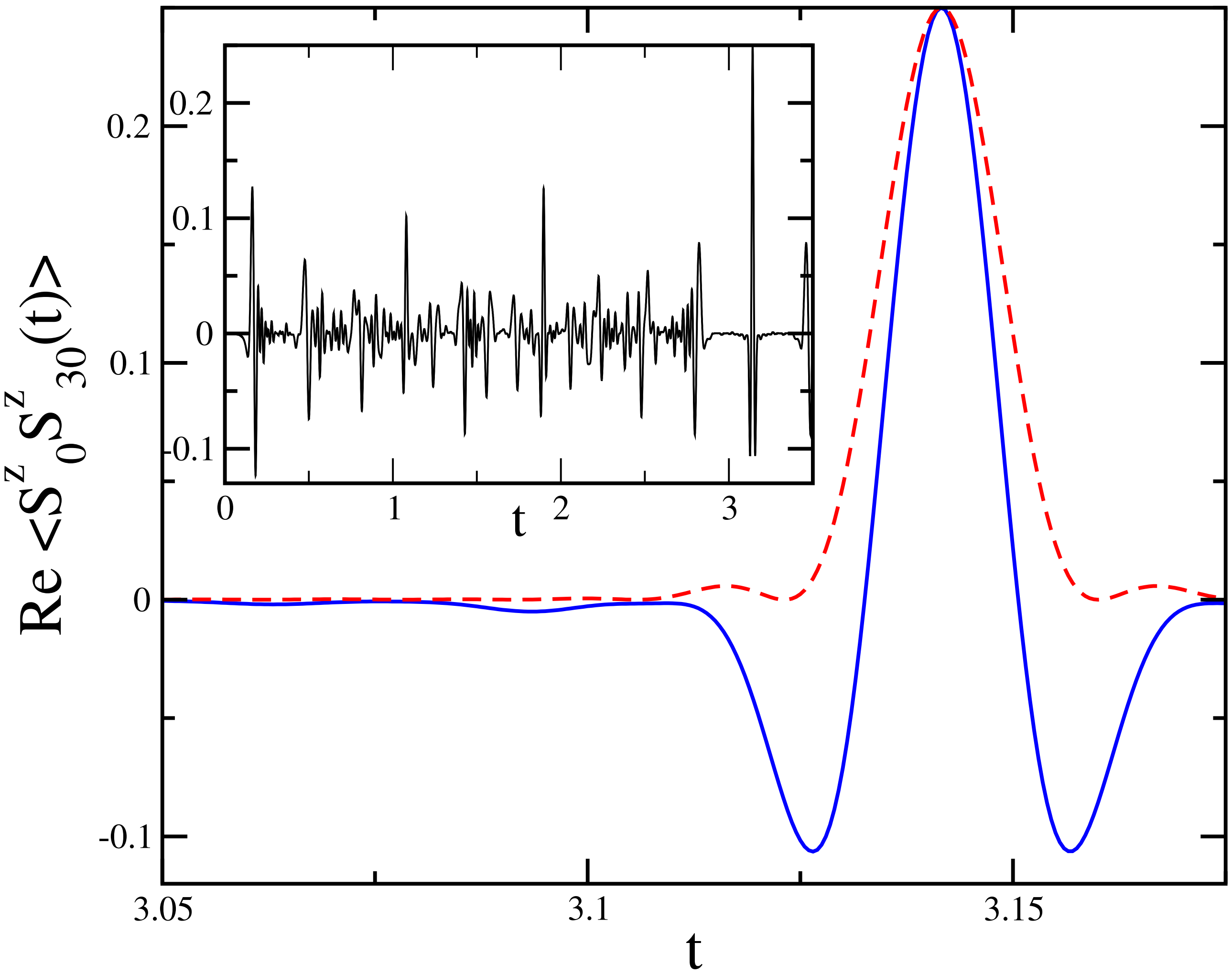
Results for a 31-spin chain:

- cosine-like dispersion
- almost constant ($\pm 3.3\%$ variation) couplings
- perfect transfer

(For 50 sites the coupling varies only by $\pm 1\%$.)

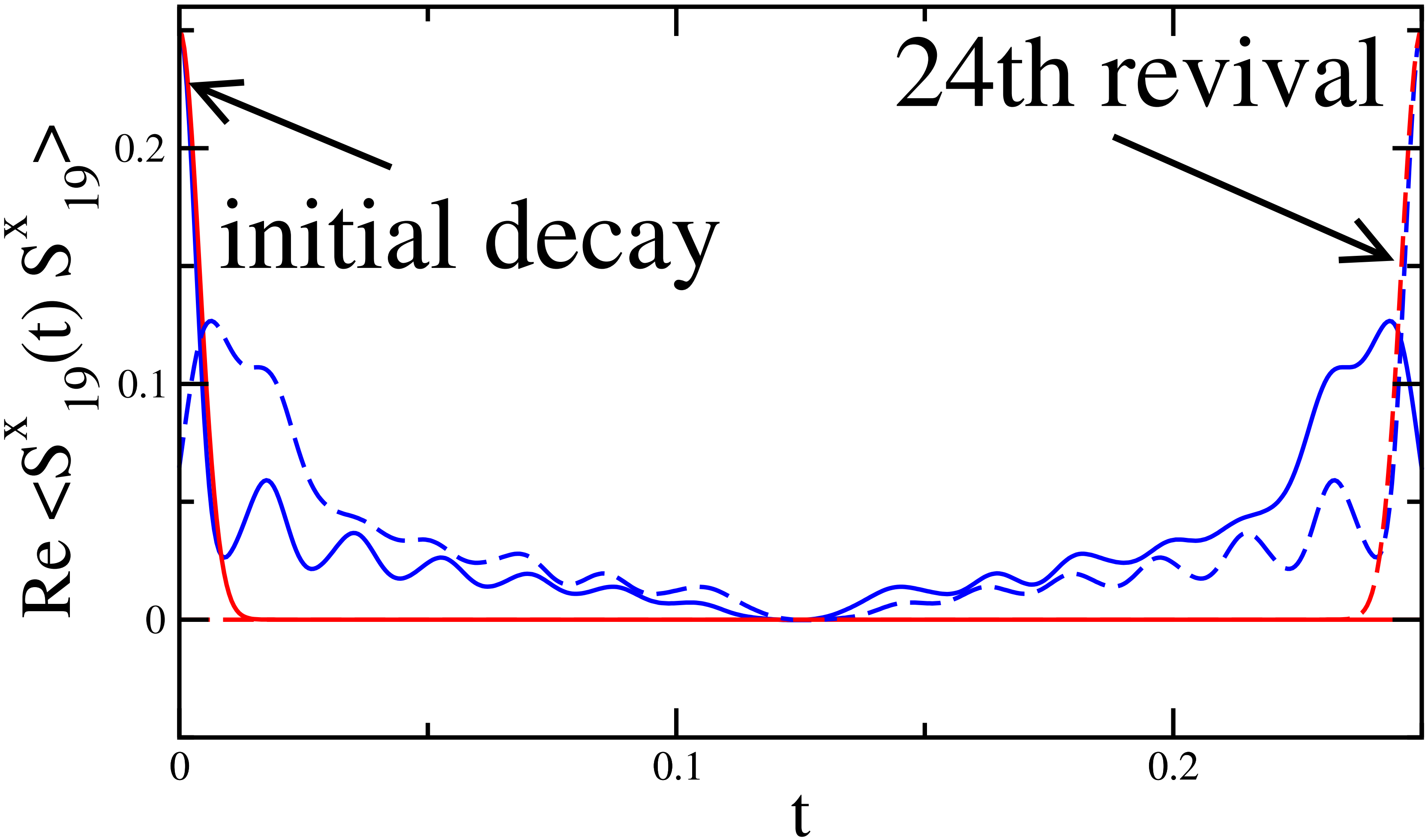


Safe transfer at any temperature



Real part of $\langle S_0^z S_{30}^z(t)\rangle$ in a 31-spin chain at $T = 0$ and $T = 1000$, near $t = \pi$. The maximum possible value $1/4$ of the correlation at $t = \pi$ demonstrates perfect state transfer.
Inset: same correlation for $T = 0$ over an extended time range shows somewhat irregular behavior at intermediate times.

Perfect long-time periodicity



Autocorrelation of the x spin component at site 19 in a 41-site chain, at times t (solid) and $t + 0.25 - 48\pi$ (dashed), at $T = 0$ and $T = 10^4$.

Jordan-Wigner $\rightarrow$ many-fermion correlation involving lattice sites 0 through 19.

Note the rapid decay and the absence of oscillations at high T. ($\rightarrow$ Gaussian).

P. Karbach, JS, Phys. Rev. A **72** 030301(R) (2005)

Conclusions

- There is an infinitely large class of inhomogeneously coupled spin chain systems capable of perfect quantum information transfer.

- The freedom of choice within that class allows for some spin chain engineering.

- Perfect state transfer over fairly long distances in a chain with almost homogeneous exchange coupling and without external magnetic field.

- In contrast to many previous proposals, there is no restriction to the transfer of single-spin states at zero temperature. The systems discussed here can transfer genuinely entangled states involving several qubits, at arbitrary temperature.

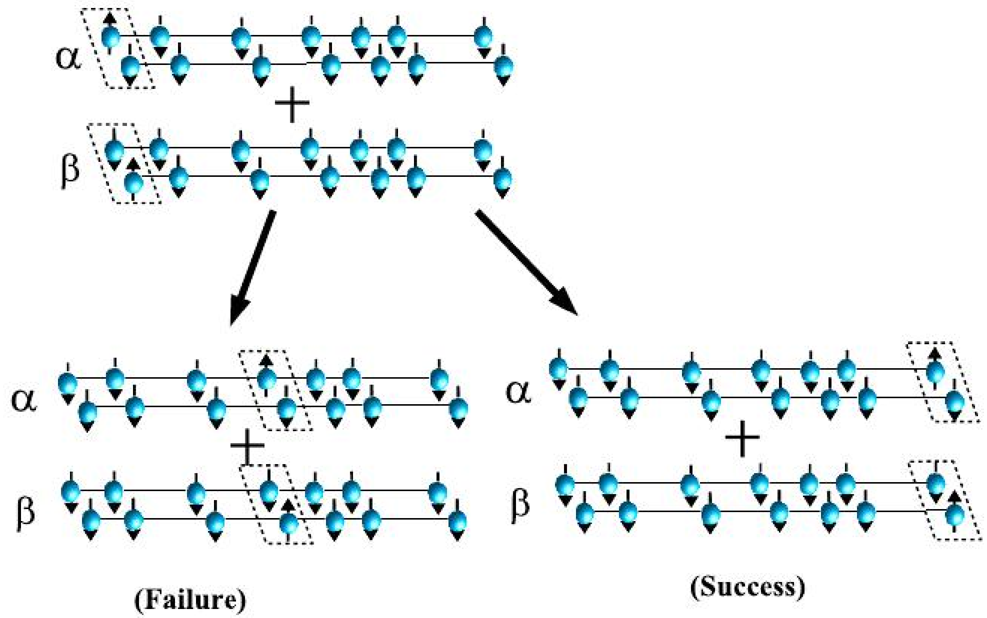
- Sensitivity to perturbations like noise and imperfections will be / has been the subject of further research.
  (A. Zwick, G.A. Alvarez, JS, O. Osenda, Phys. Rev. A **84** 022311 (2011), **85** 012318 (2012))

- General reference:
  G.M. Nikolopoulos and I. Jex (eds.), *Quantum State Transfer and Quantum Network Engineering*, Springer Series in Quantum Science and Technology (maybe 2013).

Alternative concepts and other developments in quantum information transport through spin chains

- The dual-rail protocol (D. Burgarth and S. Bose, Phys. Rev. A **71**, 052315 (2005))

  State $\alpha|0\rangle + \beta|1\rangle$ is encoded as $\alpha|\uparrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle$ by Alice, all other spins in both "rails" are $\downarrow$.
  Bob measures total magnetization of the two end spins on his side without measuring them individually. Transfer is successful when total magnetization is zero.
  Transfer is perfect if it is sucessful.
  Only single-qubit transport.



- Repeated swap to memory (V. Giovannetti and D. Burgarth, PRL **96**, 030501 (2006))
  Only single-qubit transport in a chain with all spins initially $\downarrow$.
  Bob swaps the state of his end spin with a fresh $|0\rangle$ from memory at regular intervals.
  After some time the whole chain is again $\downarrow$ and all information is in memory.

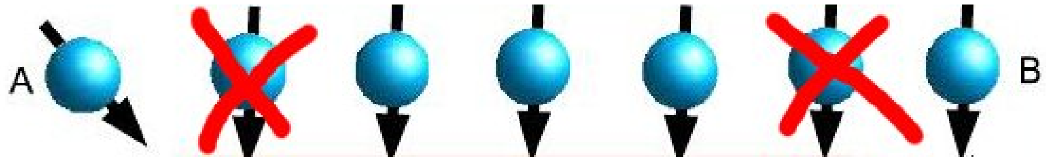
- Redundant encoding (Z.-M. Wang et al. arXiv:0812.4578)

  Encode a single logical qubit in more than one physical qubit (=spin) and study transmission properties of several possible encodings.

  Only single-qubit transport.

- "Double-hole" chain (G. Gualdi et al. arXiv:0812.2404, PRA **78**, 022325 (2008))

  Ferromagnetic long-range dipolar coupling, the second and second to last spins are eliminated. The two lowest energy eigenstates of the resulting system are equivalent to $|01\rangle \pm |10\rangle$ (for the end spins) with all higher energy states separated by a sufficiently large gap.
  Effective dynamics in this subspace allows for perfect transport.
  Only single-qubit transport.

  (Similar proposals were also made earlier by others, see the review by Bose, arXiv:0802.1224.)

- Externally pulsed Ising chain (J. Fitzsimons and J. Twamley, PRL **97**, 090502 (2006); G.A. Paz-Silva et al. PRL **102**, 020503 (2009))

  Uniformly coupled Ising chain subject to global pulses (NMR...).
  Basic idea: repeated SWAP operations between neighboring spins.
  Can also be used to mirror states involving more than one qubit.

Jordan-Wigner: The ugly details

Single-spin operators $\longrightarrow$ many-fermion operators

$$S_i^z = a_i^\dagger a_i - \frac{1}{2} = n_i - \frac{1}{2}$$

$$S_i^+ = (-1)^{\sum_{k<i} n_i} a_i^\dagger \quad ; \quad S_i^- = (-1)^{\sum_{k<i} n_i} a_i$$

$$(-1)^{a_k^\dagger a_k} = (a_k^\dagger + a_k)(a_k^\dagger - a_k)$$

$$\Longrightarrow S_i^+ = (a_1^\dagger + a_1)(a_1^\dagger - a_1)(a_2^\dagger + a_2)(a_2^\dagger - a_2) \ldots (a_{i-1}^\dagger + a_{i-1})(a_{i-1}^\dagger - a_{i-1}) a_i^\dagger$$

$\lhd$