Decoherence, semiclassical picture

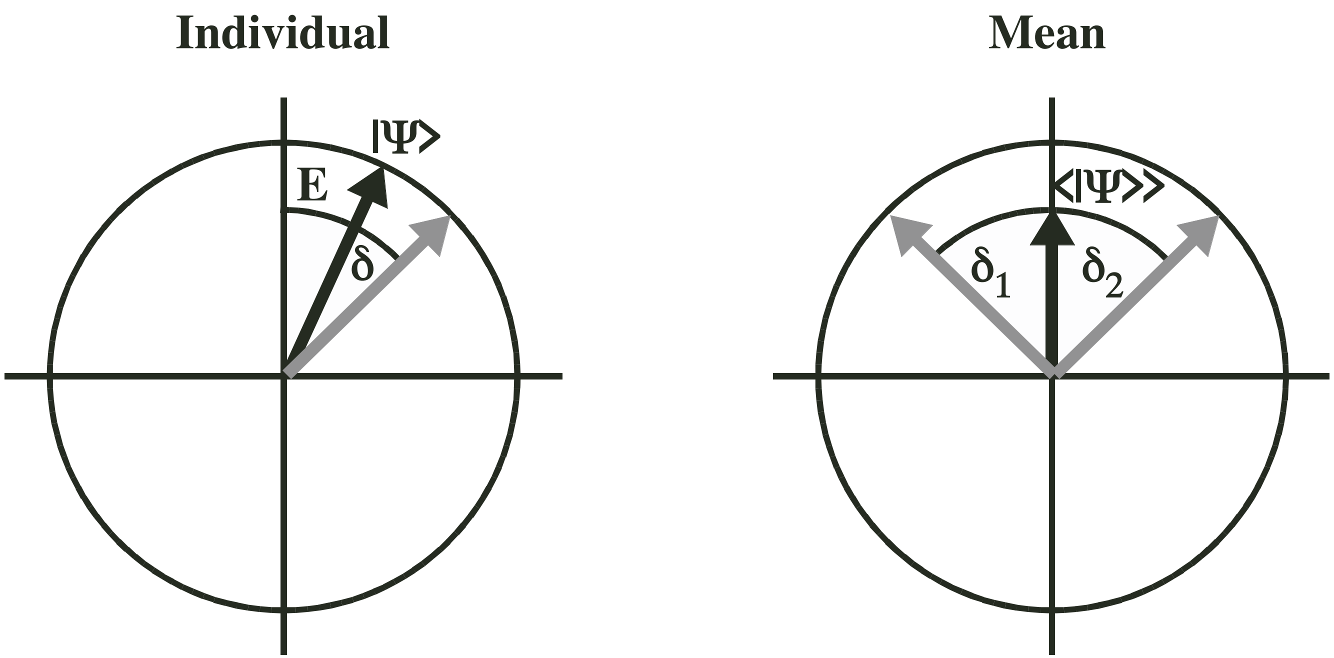
Consider a qubit in a superposition state

$$|\psi(0)\rangle = a|0\rangle + b|1\rangle.$$

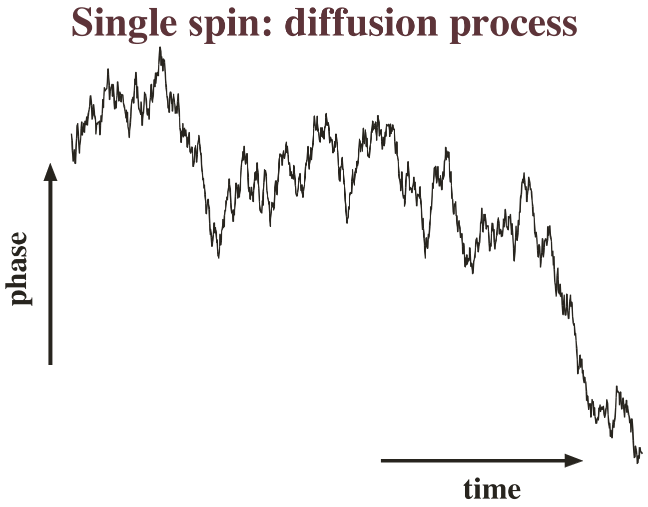
If $|0\rangle$ and $|1\rangle$ are eigenstates of the Hamiltonian:

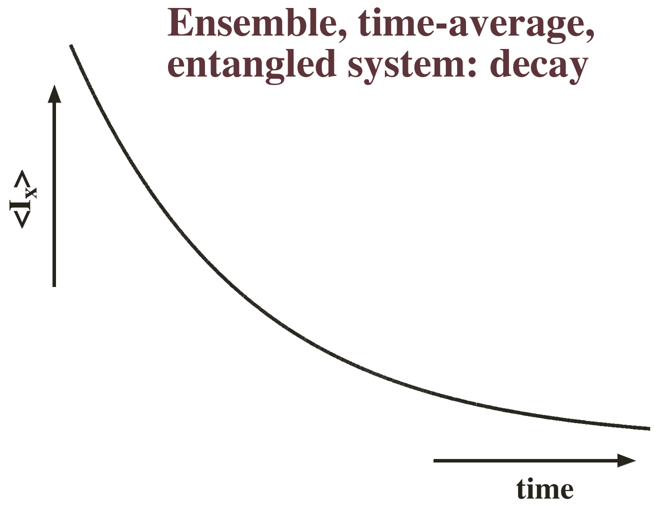
$$|\psi(t)\rangle = a|0\rangle e^{-iE_0 t/\hbar} + b|1\rangle e^{-iE_1 t/\hbar}.$$

That evolution is perturbed by additional energy contributions δ (left). In an ensemble, averaging over many random additional energy contributions leads to amplitude reduction (right) .



The phase of a single spin in a random magnetic field shows a random walk (left). Average over many independent random fields for many spins leads to signal decay by dephasing (right).





Phenomenologically, dephasing is often assumed to be exponential:

$$\rho_{ij}(t) = \rho_{ij}(0) e^{-i(E_i - E_j)t/\hbar} e^{-t/T_2}.$$

The decay of diagonal density operator elements (populations) is described by a different time constant, the longitudinal relaxation time T_1:

$$\rho_{ii}(t) = \rho_{ii}(0) e^{-t/T_1}.$$

Decoherence, by interaction with a bath

Two qubits: $\vec{S}_1$ (system) and $\vec{S}_2$ (bath), with exchange interaction

$$\mathbf{H} = \frac{\omega}{\hbar} \vec{S}_1 \cdot \vec{S}_2$$

with an initial product state

$$|\psi(0)\rangle = \Big(a|\uparrow\rangle + b|\downarrow\rangle \Big)_1 \otimes \Big(c|\uparrow\rangle + d|\downarrow\rangle \Big)_2$$

evolves into

$$\begin{aligned} \exp\left(\frac{i\omega t}{4}\right) |\psi(t)\rangle =& ac|\uparrow\uparrow\rangle + bd|\downarrow\downarrow\rangle \\ &+ \frac{1}{2}[ad(1 + e^{i\omega t}) + bc(1 - e^{i\omega t})]|\uparrow\downarrow\rangle \\ &+ \frac{1}{2}[ad(1 - e^{i\omega t}) + bc(1 + e^{i\omega t})]|\downarrow\uparrow\rangle. \end{aligned}$$

Periodic and harmonic time evolution. The initially non-entangled state becomes entangled (and non-entangled again periodically) as witnessed by the concurrence

$$C = 2|\alpha\delta - \beta\gamma| \text{ for the 2-qubit state } \alpha|\uparrow\uparrow\rangle + \beta|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle + \delta|\downarrow\downarrow\rangle$$

$C = 0$ for a product state, $C = 1$ for a maximally entangled (e.g. Bell) state; for "our" state C is strictly periodic

$$C = |ad - bc|^2 |\sin \omega t|.$$

A very simple special case is

$$b = c = 0, a = d = 1, \text{ so that } |\psi(0)\rangle = |\uparrow\downarrow\rangle$$

which becomes maximally entangled for $t = \frac{\pi}{2\omega}$

$$\exp\left(\frac{i\pi}{8}\right) |\psi(\frac{\pi}{2\omega})\rangle = \frac{1+i}{2}|\uparrow\downarrow\rangle + \frac{1-i}{2}|\downarrow\uparrow\rangle.$$

The reduced density operator for the system then becomes

$$\boldsymbol{\rho}_1\left(\frac{\pi}{2\omega}\right) = \mathrm{Tr}_2 \left|\psi\left(\frac{\pi}{2\omega}\right)\right\rangle\left\langle\psi\left(\frac{\pi}{2\omega}\right)\right| = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

maximally mixed, even though the initial state was pure: information (about the relative phases between the system basis states) has been transferred into the bath.

Error correction basics

Errors are unavoidable $\Longrightarrow$ error correction is necessary in quantum as well as in classical computing.

Two-step process:
1)error detection
2)error correction

Step 1) involves measurement which affects the information. Makes quantum case problematic. One of the key ideas of quantum error correction schemes: try to separate information about possible errors (which is measured) from "useful" information (which is corrected).

Threshold theorem: Quantum computations of arbitrary length may be performed reliably, provided that the error probability per step is below a certain threshold.

The threshold value depends on

- computer model
- error model.

Example: Network model; every gate works perfectly with probability $1 - \epsilon$ and fails (that is, performs an arbitrary norm-conserving operation) with probability ϵ. Assuming the application of certain error-correcting codes (see below), it can be shown that for $\epsilon < 2.73 \cdot 10^{-5}$ everything is (rigorously) fine. (P. Aliferis, D. Gottesman, and J. Preskill, Quantum Information and Computation **6**, 97-165 (2006), see also P. Aliferis, Thesis, CalTech 2007. Both available on the ArXive.)

Numerical exidence exists that the error bound in fact is about three orders of magnitude larger. (E. Knill, Nature **434**, 39-45 (2005))

Classical error correction

Redundancy is the key: if part of the information is lost or deformed it can be reconstructed.

Simplest example: transmission of single bits.

Every classical bit 0 or 1 is transmitted correctly with probability $1-p$, and flipped with probability $p \ll 1$ (multiple flips neglected).

Achieve redundancy by mapping logical bit values to larger physical bits (code words), for example

$$0_L \mapsto 000, \qquad 1_L \mapsto 111$$

Probabilities...
all three bits transmitted correctly $\rightarrow (1-p)^3$
only one bit flipped $\rightarrow 3p(1-p)^2$
two or three bits flipped $\rightarrow p^2(3-2p)$

Error correction by majority rule: If one bit differs from the other two after transmission, flip it.
Failure probability: $\rightarrow p^2(3-2p)$.

Quantum case: no copying, no majority rule, because not all qubits can be measured, because measurement $\rightarrow$ collapse: $a|0\rangle + b|1\rangle \rightarrow |0\rangle$ or $|1\rangle$.

Quantum error correction

- Decide which classes of errors should be corrected.

- Measure which kind of error has occurred and store that information (the error syndrome) in additional (ancillary) qubits ($\rightarrow$ redundancy), without changing the original quantum information.

- Correct any errors detected in the way suggested by the error syndrome measured.

Additional kinds of errors to be corrected $\rightarrow$ additional overhead (ancillary qubits, correction operations) needed.

Note that qubits are subject to a continuum of possible errors, whereas classical bits may just be flipped.

Guiding principle of quantum error correction: Embed the qubit Hilbert space in a larger one in such a way that no conceivable error maps the state representing $|0\rangle$ to the one representing $|1\rangle$ or vice versa. Then try to detect inadmissible components and eliminate them.

Single spin-flip error

Transmission of qubits between Alice and Bob. Action of the transmission channel:

$\mathbf{1}$ with probability $1 - p$; $\mathbf{X}$ with probability p

Alice cannot use copying for error detection, but she can use entanglement in order to transmit the state $a|0\rangle + b|1\rangle$.

Alice initializes two ancillary qubits in the state $|0\rangle; \Longrightarrow$ initial state of the three qubits is

$$|\psi_0\rangle = a|000\rangle + b|100\rangle.$$

Next she applies CNOT(1,2) and CNOT(1,3) $\Longrightarrow$

$$|\psi_1\rangle = a|000\rangle + b|111\rangle.$$

NB: This operation is not cloning: cloning (if it were possible) would lead to a product state of the three qubits with all of them in the same single-qubit state. Finally Alice sends the three qubits down the faulty channel, and relaxes.

Probabilities...

all three qubits transmitted correctly $\rightarrow (1-p)^3$

only one qubit flipped $\rightarrow 3p(1-p)^2$

two qubits flipped $\rightarrow 3p^2(1-p)$

three qubits flipped $\rightarrow p^3$

In the last case error detection is impossible since Bob has received a combination of the legal "quantum code words" $|000\rangle$ and $|111\rangle$. Bob always gets a superposition of two complementary states: if qubit 2 was flipped during transmission, Bob receives instead of $|\psi_1\rangle$ the state

$$|\tilde{\psi}_1\rangle = a|010\rangle + b|101\rangle.$$

$\Longrightarrow$ measurement of $\mathbf{Z}_1\mathbf{Z}_2$ yields the same value (here -1) for both components of Bob's state; same for $\mathbf{Z}_1\mathbf{Z}_3$. Bob's state is thus always an eigenstate of $\mathbf{Z}_1\mathbf{Z}_2$ and $\mathbf{Z}_1\mathbf{Z}_3$. Errors can be detected and identified by measuring $\mathbf{Z}_1\mathbf{Z}_2$ and $\mathbf{Z}_1\mathbf{Z}_3$, and they can be corrected.

In our example $\mathbf{Z}_1\mathbf{Z}_2 = -1$ and $\mathbf{Z}_1\mathbf{Z}_3 = 1 \Longrightarrow$ qubit 2 has been flipped. Bob applies $\mathbf{X}_2$ to restore the state $|\psi_1\rangle$, apart from a sign. Similar if qubits 1 or 3 have been flipped.

Failure if two qubits are flipped: the state $a|101\rangle + b|010\rangle$ yields the same values for $\mathbf{Z}_1\mathbf{Z}_2$ and $\mathbf{Z}_1\mathbf{Z}_3$ as the state $|\tilde{\psi}_1\rangle$ and is "corrected" to $a|111\rangle + b|000\rangle$.

Bob can reconstruct Alice's original single-qubit state by repeating (=reversing) Alice's first two CNOT operations (with qubit 1 as control and qubits 2 and 3 as targets, respectively). Probability of failure: $\mathcal{O}(p^2)$, as compared to $\mathcal{O}(p)$ without error correction.

Continuous phase errors

Example of a "continuous" type of error: a random z axis rotation given by

$$\mathbf{P}(\varepsilon) = e^{i\varepsilon\phi\mathbf{Z}} = \begin{pmatrix} e^{i\varepsilon\phi} & 0 \\ 0 & e^{-i\varepsilon\phi} \end{pmatrix} = \cos(\varepsilon\phi)\mathbf{1} + i\sin(\varepsilon\phi)\mathbf{Z}.$$

This acts like a combination of no error and a "phase flip" caused by the operator $\mathbf{Z}$:

$$\mathbf{Z}(a|0\rangle + b|1\rangle) = a|0\rangle - b|1\rangle.$$

Consider the action of $\mathbf{Z}$ in the basis of the eigenstates $|+\rangle$ and $|-\rangle$ of $\mathbf{X}$:

$$|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \quad ; \quad \mathbf{X}|\pm\rangle = \pm|\pm\rangle :$$

obviously

$$\mathbf{Z}|\pm\rangle = |\mp\rangle,$$

that is, $\mathbf{Z}$ causes a bit flip in the basis given by the eigenstates of $\mathbf{X}$, and we have already seen how a bit flip can be corrected for.

General single qubit errors

The most general single-qubit error: a general unitary 2×2 matrix, combined with a projection to some axis. Can be written in terms of $\mathbf{1}, \mathbf{X}, \mathbf{Y}$, and $\mathbf{Z}$. Errors caused by $\mathbf{X}$ and $\mathbf{Z}$ can be corrected for by simple procedures, and also those caused by $-i\mathbf{ZX} = \mathbf{Y}$.

Shor's 1995 nine-qubit code combines the two procedures just discussed. Main idea: concatenation of two redundant codes: the original logical qubit is redundantly encoded in three qubits in order to fight one kind of error, and then each of these three qubits is again encoded in three qubits to take care of the second type of error.

Alice first appends two additional qubits initialized to the state $|0\rangle$ and then applies $\mathbf{H}_3\mathbf{H}_2\mathbf{H}_1$ CNOT $(1,3)$ CNOT $(1,2)$ which yields

$$|0\rangle \rightarrow |+++\rangle \quad ; \quad |1\rangle \rightarrow |---\rangle.$$

Step 2: Alice adds two fresh $|0\rangle$ qubits to each of the three code qubits and applies the two CNOTs again. Result: one logical encoded in entangled states of nine physical qubits

$$\begin{aligned} |0\rangle &\rightarrow \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ |1\rangle &\rightarrow \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle). \end{aligned}$$

Arbitrary single-qubit errors may be detected and corrected:

- Single qubit flip in the first triplet: detect by $\mathbf{Z}_1\mathbf{Z}_2$ and $\mathbf{Z}_1\mathbf{Z}_3$, correct by the appropriate $\mathbf{X}$.
- Phase flip on one qubit of the first triplet: changes $|000\rangle+|111\rangle$ to $|000\rangle-|111\rangle$. Detect by *comparing* the signs in the three-qubit blocks. Note: $\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3$ maps $|000\rangle \rightarrow |111\rangle$ and vice versa $\rightarrow$ sign. Sign comparisons between blocks are thus performed by $(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3)(\mathbf{X}_4\mathbf{X}_5\mathbf{X}_6)$ and $(\mathbf{X}_4\mathbf{X}_5\mathbf{X}_6)(\mathbf{X}_7\mathbf{X}_8\mathbf{X}_9)$. A phase flip on any of the first three qubits can then be repaired by applying $\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3$.
- Similar for $\mathbf{ZX} = i\mathbf{Y}$.

An entire continuum of arbitrary single qubit errors is kept under control by really taking care of only a finite (and very small) set of errors!

There are codes providing the same degree of protection with 7 and even 5 physical qubits per logical qubit, but with more complicated operations necessary.

Stabilizer codes: a group-theoretical concept.
The Pauli matrices plus the unit matrix, with prefactors ± 1 and $\pm i$ form a group: the Pauli group, which can be generalized to n qubits. If we can find a subgroup S of the n-qubit Pauli group and a set V_S of n qubit states that is invariant under the action of S, then S is called a stabilizer and V_S is the vector space stabilized by S.

Basis vectors of V_S can be used as codewords: any legal combination of codewords is mapped to another legal combination by elements of S, and errors can be detected since they are mapped to something else.